# Noise and photon statistics

Thinking of light as an electromagnetic wave provides a solid foundation for modeling diffraction, interference, polarization, and for using Fourier analysis. It even works well for correlation, as demonstrated profoundly in the experiments and theory of Hanbury Brown. [23, 24] However, perhaps more fundmentally we think of light as a photon with no charge or mass, traveling at speed $c$, carrying an energy $E = hf$, and possessing an integer spin of $\hbar$. This framework is equally effective for discussing the topics we have addressed, but in a different formalism yielding the same results in the classical limit. However when it comes to very weak signals and detection through the photoelectric effect, we count single photons discretely, and time their arrival precisely, while in the same experiment sort them by "wavelength", *i.e.* energy, and "polarization", *i.e.* spin. A common problem that arises is to describe the statistics of photons and the resultant photoelectrons, and to understand the response of devices that detect them and are used to form images of a scene in which spatial direction, arrival time, color and measurement uncertainty are important.

Fried [37] addressed this issue directly by considering the statistics of photoelectrons based on a conventional assumption that photons obey Poisson statistics. Previously we have used the Central Limit Theorem to invoke a Gaussian distribution to describe the statistics of light from fluctuations, and these two approaches are related but not the same. There is an intrinsic statistical character of light from thermal sources, and there is also a statistical character imposed on a signal by processes either in the source, the medium between it and the detector, or the detector itself. Indeed, the analysis of the detector statistics led Fried to look closely at how efficiency in the detector affected the noise in the photoelectron signal. Here we will ask fundamentally what is the best statistical distribution to use to describe light from a thermal source, review Fried's analysis of the signal-to-noise ratio, and describe a method of determing the properties of a detector from a measurement of the noise in the photon signal it produces.

## Poisson distribution

The starting point is the probability distribution of finding $n$ photons in a measurement when the average number over many measurements is $\langle n \rangle$. Following Fried we use the Poisson distribution

$$P(n; \langle n \rangle) = \frac{\langle n \rangle^n}{n!} \exp(-\langle n \rangle) \ . \tag{152}$$

Fox [38] offers a justification that also aids understanding the limitations of the Poisson distribution. He considers a coherent beam of light with a steady flux, and divides a length of it containing $\langle n \rangle$ photons into $N$ segments. The probability of finding a photon within one of these parts is $p = \langle n \rangle / N$, and when $N$ is large enough the chance of finding more than one photon there is negligible compared to the chance of finding only one. We test each of the $N$ bins to see if there is a photon in it. The probability of finding $n$ bins with a photon and $(N - n)$ with none is probabilityh of of getting success in $n$ out of $N$ trials. That is, it is the binomial distribution

$$P(n;N) = \frac{N!}{n!(N-n)!}\, p^n\, (1-p)^{N-n} \tag{153}$$

Substituting for $p$ we have

$$P(n;N) = \frac{N!}{n!(N-n)!}\, (\frac{\langle n\rangle}{N})^n\, (1-\frac{\langle n\rangle}{N})^{N-n} \tag{154}$$

To assure that there is only one photon in a bin we factor out the $N$ terms and take the limit $N \to \infty$ of

$$P(n;N) = \frac{1}{n!}\frac{N!}{(N-n)!\, N^n}\, \langle n\rangle^n\, (1-\frac{\langle n\rangle}{N})^{N-n} \tag{155}$$

The fraction in the multiplier

$$\frac{1}{n!}\frac{N!}{(N-n)!\, N^n}$$

is found by considering its logarithm

$$\ln(\frac{1}{n!}\frac{N!}{(N-n)!\, N^n})$$

using the Stirling approximation

$$\lim_{N\to\infty}(\ln N!) = N\ln N - N$$

$$\lim_{N\to\infty}\ln(\frac{1}{n!}\frac{N!}{(N-n)!\, N^n}) = 0$$

to show that it is

$$\lim_{N\to\infty}\frac{1}{n!}\frac{N!}{(N-n)!\, N^n} = 1$$

The power term expands

$$(1-\frac{\langle n\rangle}{N})^{N-n} = 1-(N-n)(\frac{\langle n\rangle}{N})^1 + +\frac{1}{2!}(N-n)(N-n-1)(\frac{\langle n\rangle}{N})^2 + \cdots$$

and taken to the limit becomes

$$1-\langle n\rangle + \frac{1}{2!}\langle n\rangle^2 - \cdots = \exp(-\langle n\rangle)$$

Combining these two terms in the limit of large $N$ yields the Poisson statistics for detecting photons in a coherent beam

$$P(n;\langle n\rangle) = \frac{\langle n\rangle^n}{n!}\exp(-\langle n\rangle) \tag{156}$$

The variance of the Poisson distribution is

$$\sigma_n^2 = \langle (n - \langle n \rangle)^2 P(n; \langle n \rangle) \rangle \tag{157}$$

which can be shown for the Poisson distribution to be

$$\sigma_n^2 = \langle n \rangle \tag{158}$$

The standard deviation $\sigma$ in $n$ is therefore

$$\sigma_n = \sqrt{\langle n \rangle} \tag{159}$$

This means that if we detect $n$ photons there will be a Poisson distribution about that value and in various trials we will measure a standard deviation of $\sigma_n$ as intrinsic shot noise in the signal. The resulting signal-to-noise ratio is

$$\text{SNR} = \langle n \rangle / \sqrt{\langle n \rangle} \tag{160}$$

$$= \sqrt{\langle n \rangle} \tag{161}$$

Poisson statistics applies to coherent light which has a perfectly stable flux lasting forever. Light which has more noise is deemed *super-Poissonian* and includes thermal radiation and partially coherent light. One contribution is a consequence of Bose-Einstein statistics of photons. *Sub-Poissonian* light has less noise and would result from photons which are more ordered in time, that is photons that do not have the uniformly random behavior that led to the Poisson distribution. [38]

## Signal-to-noise ratio

Fried [37] considered the Poisson distribution of both the photons and the photoelectrons produced by a real detector. He demonstrated that if a detector had quantum efficiency $\eta$ and produced $\eta n$ photoelectrons for $n$ incident photons, the signal-to-noise ratio measured will be that of the detected photons. That is, he found that

$$\text{SNR} = s/\sigma_s \tag{162}$$

$$= \eta n / \sqrt{\eta n} \tag{163}$$

$$= \sqrt{\eta n} \tag{164}$$

His conclusion was that a Poisson distribution of the electrons produced in detection of light by the photoelectric effect contained all the photon and photoelectron noise that should be considered.

There are, however, additional sources of noise in the measurement process that should be considered independently, and these have been examined in the astronomical literature where their analysis is critical to understanding uncertainty in faint object and variability detection. Howell [39] found that in 2-dimensional measurments the signal-to-noise ratio should include the shot noise we have found here and noise from the sky background, dark signal (which is subtracted from the data) and read noise. The background and dark signals add as if they are part of the signal to be measured in the determination of a total shot noise. The read noise, which is attributed to electronics and is independent of the analog noise sources considered by Fried, is added in an approximation to quadrature. The estimation of contributions from methods of removing background from signal is well-studied and verified by experiments. While the addition of these noise sources to photon statistics and the importance of sensor read-noise are not as well studied, the procedure with modifications is now widely followed in practice. An example is given by Collins *et al.* [40] for aperture photometry with a charged coupled device where the total noise is estimated by

$$\sigma_{total} = \frac{\sqrt{GF_\star + n_{pix}(1 + \frac{n_{pix}}{n_b})(GF_S + F_D + F_R^2 + G^2\sigma_f^2)}}{G} \tag{165}$$

The "gain" in electrons/digital unit is $G$ , $F_\star$ is the net integrated digital counts in the aperture, $n_{pix}$ is the number of pixels in the aperture, $n_b$ is the number of pixels used for the sky background subtraction, $F_S$ is the sky background counts per pixel in digital units, $F_D$ is the total dark electrons (not counts) per pixel, $F_R$ is the read noise in electrons/pixel, and $\sigma_f$ is the standard deviation of a fractional count lost to digitization in comparison apertures. Clearly one factor we need to understand better is the so-called gain, which provides the calibration between the digital output of a sensor system and in the incident photons that are detected (after allowing for quantum efficiency). Gain in this sense is given in units of electrons/ADU where "ADU" means analog-to-digital unit. It is better thought of as inverse gain, telling us how much signal we get for each photon that creates a photoelectron.

## Photoelectron gain

In principle the determination of the gain parameter $G$ described above for a detection of light with a photoelectron device such as a CCD sensor could be found by illuminating the sensor with a known flux, allowing for quantum efficiency, and comparing these data with the actual measurements. There are obvious problems with doing this in practice, not the least of which is the lack of suitable standard flux lamps with spectrally limited bandpasses, and the separation of gain from quantum efficiency. Another method that is more practical is to measure the noise in the detected signal.

The key to this procedure is that the noise is characteristic of the photons that are detected as described by the $\eta n$ term in Fried's Equation 164. This means that for a signal $s = \eta n/G$, the actual noise is still determined by $\sqrt{\eta n}$ and that the signal-to-noise ratio must still be

$$s/\sigma_s = \sqrt{\eta n} \tag{166}$$

However, we do not measure the right hand side directly, but rather find $s$, from which

$$s/\sigma_s = \sqrt{Gs} \tag{167}$$

$$s^2/\sigma_s^2 = Gs \tag{168}$$

$$\sigma_s^2/s = 1/G \tag{169}$$

The right and side is independent of flux, and the left hand side is directly measurable for any flux to which the detector will respond. That is

$$1/G = \sigma_s^2/s \tag{170}$$

Of course at low levels of illumination the right hand side is affected by read noise as well as by other measurement issues such as described for aperture photometry above. If those factors are constant and represented by $\sigma_r$ in ADU, while $\sigma_{total}$ is what we measure in total, the actual signal noise is estimated from

$$\sigma_s^2 = \sigma_{total}^2 - \sigma_r^2 \tag{171}$$

from which we have

$$1/G = \sigma_{total}^2/s - \sigma_r^2/s \tag{172}$$

$$\sqrt{1/G} \approx \sigma_{total}/\sqrt{s} \tag{173}$$

This leads to the expectation that simply looking at the asymptote as $s$ is increased will suppress the read noise term and provide a direct measurement of inverse gain $1/G$ from the observed noise alone. Application of a model for all noise sources would of course improve the extraction of G, and possibly other detector parameters, from the noise measured as a function of signal. This is improved by performing the measurement over many examples, either by varying S, or by repeatedly sampling the data. For example, a sensor with $1024 \times 1024 = 1\,048\,576$ pixels could be exposed for 100 frames to determine $\sigma_s$. The global gain G applies to every pixel, and each one is an independent measurement.

Figure 1 illustrates this for two commercial cameras. The Mako camera using a Sony sensor shows $\sqrt{1/G} = \sigma/\sqrt{s} \to 0.35$, from which we have $G = (1./0.35)^2 = 8.2$, while the GE680 camera with an On-Semi sensor $\sqrt{1/G} = \sigma/\sqrt{s} \to 0.65$, from which we have $G = (1./0.65)^2 = 2.4$. Typically the manufacturers set the gain so that the full range of the ADC matches the well depth of the sensor. These cameras operate with 12-bit digitization for which the maximum signal is 4095. A gain of 8.2 implies that the full signal would correspond to $4095 \times 8.2 = 33579$ photoelectrons, and provide a signal-to-noise ratio of 183:1. On the other hand, a gain of 2.4 would allow fewer photoelectrons and have a lower limiting signal-to-noise of 99:1.

# Related topics

These topics are being developed for additional treatment.

- Seeing and turbulence
- Time tagged photon correlation

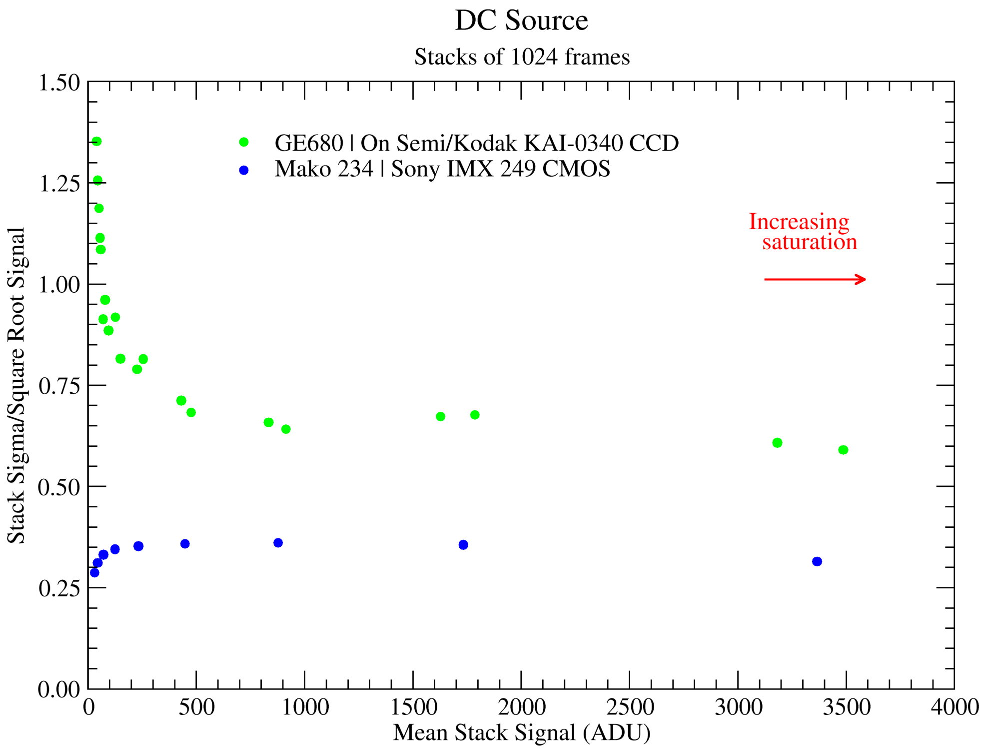


Figure 1: Noise measurement in a CCD camera. The Allied Vision Mako 234 camera uses a low-noise Sony CMOS sensor and exhibits $G \approx 8.4$ e/ADU, while the GE680 camera uses an On-Semi sensor with $G \approx 2.4$ e/ADU.