CCD Sensor Gain and Non-Linearity

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Normal Distribution

A normal distribution is a Gaussian of unit area with mean μ, standard deviation σ, and variance σ^2

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{1}$$

The standard deviation if x is square root of the mean square deviation over the distribution

$$\sigma = \sqrt{\langle (x-\mu)^2 \rangle} \tag{2}$$

The distribution one σ from its center has a value that is $e^{-1/2} = 0.6065$ of the peak. The half width at half maximum is such that

$$\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \frac{1}{2} \tag{3}$$

This leads to

$$(x-\mu)^2 = 2\ln 2\,\sigma^2 \tag{4}$$

$$\text{hwhm} = \sqrt{2\ln 2}\,\sigma \tag{5}$$

Poisson Distribution

Thinking of light as an electromagnetic wave provides a solid foundation for modeling diffraction, interference, polarization, and for using Fourier analysis. It even works well for correlation, as demonstrated profoundly in the experiments and theory of Hanbury Brown. [1, 2] However, perhaps more fundamentally we think of light as a photon with no charge or mass, traveling at speed c, carrying an energy $E = hf$, and possessing an integer spin of $\hbar$. This framework is equally effective for discussing the topics we have addressed, but in a different formalism yielding the same results in the classical limit. However when it comes to very weak signals and detection through the photoelectric effect, we count single photons discretely, and time their arrival precisely, while in the same experiment sort them by "wavelength", *i.e.* energy, and "polarization", *i.e.* spin. A common problem that arises is to describe the statistics of photons and the resultant photoelectrons, and to understand the response of devices that detect them and are used to form images of a scene in which spatial direction, arrival time, color and measurement uncertainty are important.

Fried [3] addressed this issue directly by considering the statistics of photoelectrons based on a conventional assumption that photons obey Poisson statistics. Previously we have used the Central Limit Theorem to invoke a Gaussian distribution to describe the statistics of light from fluctuations, and these two approaches are related but not the same. There is an intrinsic statistical character of light from thermal sources, and there is also a statistical character imposed on a signal by processes either in the source, the medium between it and the detector, or the detector itself. Indeed, the analysis of the detector statistics led Fried to look closely at how efficiency in the detector affected the noise in the photoelectron signal. Here we will ask fundamentally what is the best statistical distribution to use to describe light from a thermal source, review Fried's analysis of the signal-to-noise ratio, and describe a method of determining the properties of a detector from a measurement of the noise in the photon signal it produces.

The variance of the Poisson distribution is

$$\sigma_n^2 = \langle (n - \langle n \rangle)^2 P(n; \langle n \rangle) \rangle \tag{6}$$

which can be shown for the Poisson distribution to be

$$\sigma_n^2 = \langle n \rangle \tag{7}$$

The standard deviation σ in n is therefore

$$\sigma_n = \sqrt{\langle n \rangle} \tag{8}$$

This means that if we detect n photons there will be a Poisson distribution about that value and in various trials we will measure a standard deviation of σ_n as intrinsic shot noise in the signal. The resulting signal-to-noise ratio is

$$\begin{aligned} \text{SNR} &= \langle n \rangle / \sqrt{\langle n \rangle} & (9) \\ &= \sqrt{\langle n \rangle} & (10) \end{aligned}$$

Poisson statistics applies to coherent light which has a perfectly stable flux lasting forever. Light which has more noise is deemed *super-Poissonian* and includes thermal radiation and partially coherent light. One contribution is a consequence of Bose-Einstein statistics of photons. *Sub-Poissonian* light has less noise and would result from photons which are more ordered in time, that is photons that do not have the uniformly random behavior that led to the Poisson distribution. [4]

Gain

Let N_P be the number of photons in an exposure, and N_D be the "digital count", that is, the result of converting the analog signal of photoelectrons to a digital unit. We refer to the units of N_D as "analog-to-digital units" or ADU. These are related by a factor termed the "gain", g,

$$N_P = gN_D \tag{11}$$

Gain defined this way is in units of electrons per analog-to-digital unit ($e^-/(ADU)$. In the electronic sense, it is an inverse gain for the digitization process. In order to quantitatively interpret CCD data in photometry, we must determine a value for g so that the equivalent number of photons detected can be computed from the digital count. The gain coefficient is treated as a global parameter, affecting all pixels on the device identically. In practice, this is not the case, and the local variations in the global gain are corrected pixel-by-pixel through dark and flat field processing. Also, the gain described this way does not allow for a temporal dependence (charge leaking faster in longer exposures), or a spectral dependence (gain dependence on the frequency of the light).

Explicity calculate σ_P and σ_D from measurements of the number of photons and the corresponding digital counts in this way.

$$\sigma_D = \sqrt{\langle (N_D - \overline{N_D})^2 \rangle} \tag{12}$$

$$= \sqrt{\Sigma_n (N_D - \overline{N_D})^2 / n} \tag{13}$$

The actual photons corresponding to this measurement are

$$\sigma_P = \sqrt{\langle (N_P - \overline{N_P})^2 \rangle} \tag{14}$$

$$= \sqrt{\Sigma_n (N_P - \overline{N_P})^2 / n} \tag{15}$$

$$= \sqrt{g^2 \, \Sigma_n (N_D - \overline{N_D})^2 / n} \tag{16}$$

$$\sigma_P = g\sigma_D \tag{17}$$

We can measure σ_D from data collected at various exposure times or known light levels. For each of these the standard deviation in photon count $\sigma_P = \sqrt{N_P}$. We have

$$\sigma_P = g\sigma_D \tag{18}$$

$$\sqrt{N_P} = g\sigma_D \tag{19}$$

$$\sqrt{gN_D} = g\sigma_D \tag{20}$$

$$\sigma_D = \frac{1}{\sqrt{g}} \sqrt{N_D} \tag{21}$$

When σ_D is measured experimentally for various signals N_D, a fit of its dependency on N_D should show that it increases as the square root of the signal with a coefficient that is square root of the inverse gain.

Typically in CCD amplifier design to make optimal use of its dynamic range and full well capacity, the digital signal of the full well is matched to the maximum of the ADC range. For example, if the well is $100,000$ electrons and the ADC is 16-bits (65,535), the the optimal gain is $g = 100,000/65,535 = 1.53$ e/ADU. The Apogee U16 or F16 line of cameras with an On Semi (Truesense or Kodak) KAF-16803 detector is set for a gain of this order. By contrast, a sensor with a 12-bit ADC (4095) and a $30,000$ electron well would use again of $g = 30,000/4,095 = 7.3$ to digitize the full range. The least count in this case would be 7.3 electrons, approximately the read noise of the sensor and smaller than the 173 electron Poisson noise of the full well

Non-Linearity

A CCD may respond non-linearly, that is with a usually very small quadratic term in the relationship between incident photons and detected photoelectrons. If the system is designed such that the full well capacity is greater than the ADC range there will not be an abrupt "non-linear" region, but rather a gradual increase in the significance of the quadratic component up to the ADC limit. The non-linearity is otherwise not affected by the gain, which is a conversion coefficient based on linearity. The photon signal N_P corresponding to a measured digital count N_D would be

$$N_P = gN_D \tag{22}$$

to first order. Allowing for quadratic terms after removing a zero-bias in the digital signal would give

$$N_P = g(N_D + aN_D^2) \tag{23}$$

The default positive sign on a in Equation 23 reflects the usual circumstance that the counts in each pixel at higher signal levels are lower than expected given to electrons lost from the well. Note that the gain g multiplies the total digital count after correction for non-linearity. An expression of this form allows conversion of the measured count to an incident photon flux. It would be applied to thermal dark current, and optionally to the bias. Because the quadratic coefficient is typically very small, it has little effect except on signals approaching the ADC limit as long as the first linear term is unity. The gain is applied after the non-linear correction in this formulation.

A value for a would be found by measuring the quadratic component of N_D for various known relative incident fluxes N_P, for example by changing the exposure times for a source of constant flux. This inverse relationship is not a simple quadratic, but in the limit of a small quadratic correction, a, the other terms may be neglected. Solve Equation 23 for N_D and we have exactly

$$-agN_D^2 - gN_D + N_P = 0 \tag{24}$$

$$N_D = \frac{g \pm \sqrt{g^2 + 4agN_P}}{-2ag} \tag{25}$$

$$N_D = \frac{1 \pm \sqrt{1 + 4aN_P/g}}{-2a} \tag{26}$$

Use a Taylor's Series expansion for the square root

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots \tag{27}$$

$$\sqrt{1 + 4aN_P/g} = 1 + \frac{4aN_P/g}{2} - \frac{16a^2(N_P/g)^2}{8} + \dots \tag{28}$$

Retain the second order term to obtain an approximate solution for the N_D digital counts given N_P incident photons.

$$N_D = (N_P/g) - a(N_P/g)^2 \tag{29}$$

The same quadratic coefficient appears with opposite sign in both Equation 23 for the incident photons given a digital signal, and its inverse Equation 29 for the digital signal given the incident photons. Here the gain g divides the number of incident photons N_P *before* allowing for the non-linearity. The independent variable in this sense is N_P/g, which is N_D in the linear limit of small signals.

We measure the signal for various known N_P and fit it to a quadratic in order to determine the unknown quadratic non-linear response coefficient. One technique is to use a light source that is reliably constant, and record images with increasing exposure time. The photon flux F_P is admitted to the sensor per pixel per second, so that in exposure time t there are $N_P(t) = F_P t$ incident photons creating a response N_D that would be described by

$$N_D(t) = F_P t/g - a(F_P t/g)^2 \tag{30}$$

$$N_D(t) = (F_P/g)t - a(F_P/g)^2 t^2 \tag{31}$$

If we fit an observed N_D as a function of exposure time the result will be

$$N_D = \alpha_1 t - \alpha_2 t^2 \tag{32}$$

The first term, $\alpha_1 t$, is N_P/g, the injected photoelectrons. This equation has the form we need

$$N_D = x - ax^2 \tag{33}$$

if we transform it by replacing t with x/α_1, giving

$$a = \alpha_2/\alpha_1^2 \tag{34}$$

An example is shown in Figure 1 of data taken on an Apogee CCD camera with a Kodak KAF-16403 sensor. Exposures were taken with times from 1 to 9 seconds, with 0.1 second intervals approaching saturation. Interspersed 3 second exposures monitored the light source (a AA Maglite miniature xenon-filled filament lamp at half its rated voltage on a single D-cell battery). After the data were taken, AstroImageJ was used to measure the signal integrated in a 400 pixel radius aperture with bias and dark corrections. [5] The

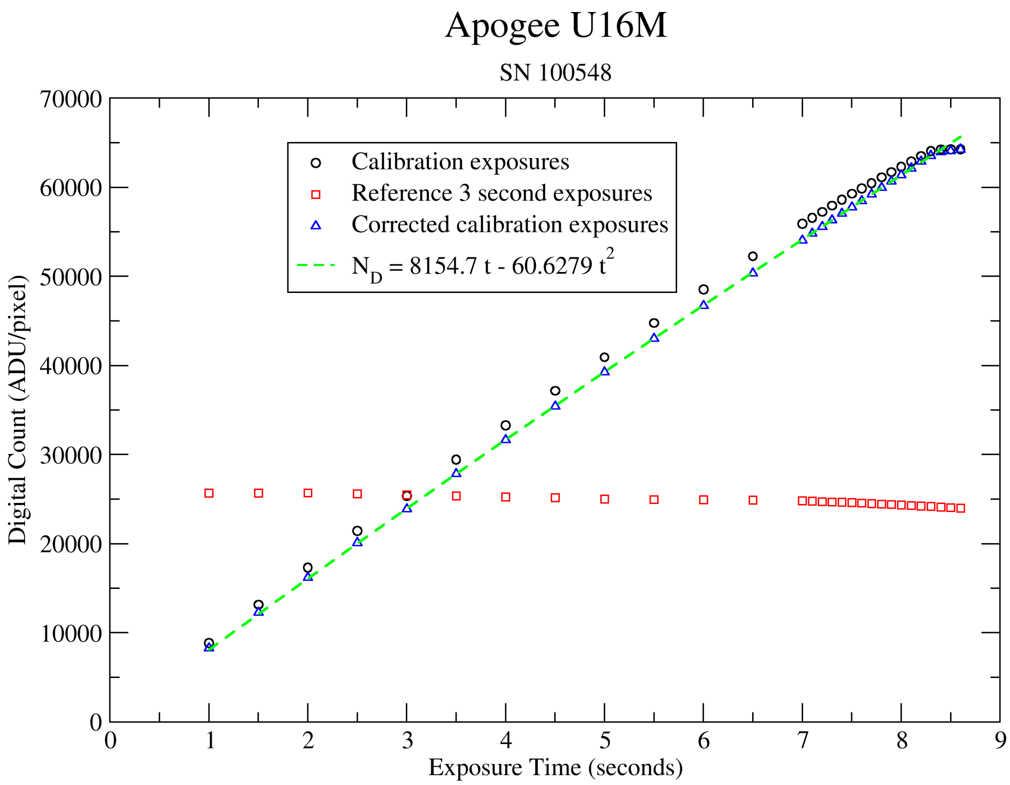

Figure 1: Fitting the non-linear response of a Kodak (On Semi) KAF-16803E CCD.

figure shows the "calibration" exposures in digital count (ADU) per pixel averaged over this aperture. The matching reference exposures also reveal a slow change in the light source output. The calibration exposures were corrected by dividing by the reference exposure and then scaling back to match the ADC saturation count. The corrected calibration exposures were fitted with a quadratic in t constrained to go through the origin since the bias had been removed. The fitting did not include the last 3 data points which are in the regime of ADC count saturation.

Here we find $\alpha_1 = 8154.7$ and $\alpha_2 = 60.6279$. Transforming to the dependence on input photoelectrons as in Equations 33 or 29, we find $a = 60.6279/8154.7^2 = 9.116 \times 10^{-7}$. The corresponding non-linear correction for data from this CCD would be given by Equation 23

$$N_P = g(N_D + aN_D^2) \tag{35}$$

where N_D is the bias and dark corrected digital count in ADU.

The experimental difficulty in this calibration process is to find a suitably constant source, which can be a vexing problem with scientific CCDs that have long readout times and for which multiple exposures would be required. However, reference exposures at a single fixed time interspersed with data exposures act as a check or may be used to correct for a slowly changing source during the analysis if needed.

The effects of uncompensated non-linearity are loss of accuracy in precision photometry, and mixing of frequencies in time-dependent analyses. The latter occurs because of the N_D^2 term, which will take components at $\cos(2\pi ft)$ and create components at $\cos(2\pi(2f)t)$, that is, a second and higher harmonic. It will also mix two different frequencies, producing the sum and difference as well as the fundamental and second harmonic of each. Since the a coefficient is usually quite small, the mixing effects are generally lost in the measurement noise. However if strong signals are measured relative to weaker standards or vice versa, the non-linear term may contribute to one signal and not to the other in proportion to the signal squared. If $a \sim 10^{-6}$, then on a CCD with large pixels where saturation is $\sim 10^5$ counts, non-linearity can approach 10% and cause very significant errors. In astronomical photometry, for example, a bright variable star may be measured with respect to a fainter one. When clouds affect both stars with the same attenuation, the resulting inferred magnitude of the brighter star can be of the order of 100 milli-magnitudes off. More typically, with changes in atmospheric transparency of the order of a factor of 2 and stars also within a factor of 10 of one another in flux, these errors are a few milli-magnitudes. Such effects may mask shallow changes in flux due to extra-solar planets if not allowed for in the analysis.

References

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[5] K. A. Collins, J. F. Kielkopf, and K. G. Stassun, "Astroimagej: Image processing and photometric extraction for ultra-precise astronomical light curves," Astronomical Journal **153**, 77 (2017).